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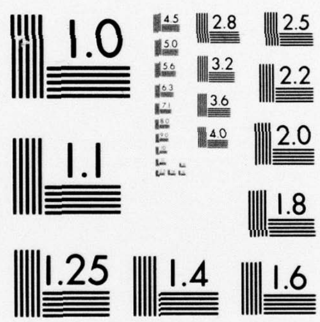

April, 1977 ESL-P-741

ROBUSTNESS AND COMPUTATIONAL ASPECTS OF NONLINEAR STOCHASTIC ESTIMATORS AND REGULATORS*

by

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ABSTRACT

Robustness properties of nonlinear extended Kalman filters with constant gains and modelling errors are presented. Sufficient conditions for the nondivergence of state estimates generated by such nonlinear estimators are given. In addition, the overall robustness and stability properties of closed-loop stochastic regulators, based upon the Linear-Quadratic-Gaussian design methodology using linearized dynamics, are presented; the sufficient conditions for closed-loop stability have a "separation-type" property.

1. INTRODUCTION

The substantial real-time computational burden imposed by the extended Kalman filter (EKF) and related suboptimal nonlinear estimators (cf. [1, Ch. 6]) significantly limits the scope of applications for which these estimators are practical. The major portion of this computational burden results from calculations associated with propagating the error-covariance matrix, which in turn is used for real-time updating of the gain matrix acting on the filter residuals. When one considers the gross nature of the approximations that are routinely made in modeling the stochastic disturbances affecting a system and to a lesser extent in modeling the interplay between these disturbances and the system's nonlinearities, it seems somewhat surprising that so much real-time computational effort should be devoted to careful propagation of the model's error-covariance matrix. The empirical fact that the EKF performs well in many applications despite the gross nature of these routine approximations suggests that perhaps the record of successes enjoyed by the EKF is attributable to an intrinsic robustness against the effects of approximations introduced in the design of its residual-gain.

With a view towards designing nonlinear estimators with greatly reduced real-time computational requirements, we have been thusly motivated to examine the possibility of employing a pre-computed approximation to the EKF residual-gain, thereby entirely eliminating the enormous computational burden of real-time error-covariance propagation. The principal implications of the results we have obtained in this connection are three-fold:

First, we have found that the real-time propagation of error-covariance may actually be unnecessary. Specifically, our results imply that for many applications one can obtain satisfactory performance from a constant-gain extended Kalman filter (CGEKF), designed to be optimal for a constant stochastic linear model crudely approximating the actual nonlinear system.

Secondly, aided by the structural simplicity of the CGEKF, we have been able to apply modern input-output techniques of analysis to prove that the CGEKF is intrinsically robust against the effects of approximations introduced in the design of its residual-gain matrix. That is, we have proved that the CGEKF design approach yields under certain conditions a nondivergent nonlinear estimator even when a relatively crude stochastic linear system model is used in designing the residual gain. These robustness results take the form of analytically verifiable conditions which also can be used to test specific CGEKF designs for nondivergence, thereby reducing the engineer's dependence on Monte Carlo simulation for design validation. Moreover, the nature of these nondivergence conditions is such as to provide a basis for the constructive modification and improvement of CGEKF designs.

Thirdly, our results combine with the linear-quadratic-optimal-regulator robustness results of [2]-[3] in a fashion reminiscent of the separation theorem of estimation and control to suggest a powerful new technique, based on linear-quadratic-Gaussian optimal feedback theory, for the synthesis of simplified dynamical output-feedback compensators for nonlinear regulator systems. The technique leads to a feedback compensator design consisting of a cascade of a CGEKF and an optimal constant linear-quadratic state-feedback (LQSF) gain matrix. We have proved that the inherent robustness of optimal linear-quadratic state-feedback against unmodeled nonlinearity [2]-[3] combines with the intrinsic robustness of the CGEKF to assure that such feedback designs will be closed-loop stable even in systems with substantial nonlinearity.

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*Research supported in part by AFOSR grant 77-3281 and in part by NASA/AMES grant NGL-22-009-124.

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This paper has been submitted to the 1977 IEEE Conference on Decision and Control and to the IEEE Transactions on Automatic Control.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) ROBUSTNESS AND COMPUTATIONAL ASPECTS OF NON-LINEAR STOCHASTIC ESTIMATORS AND REGULATORS		5. TYPE OF REPORT & PERIOD COVERED Paper
		6. PERFORMING ORG. REPORT NUMBER (14) ESL-P-741
7. AUTHOR(s) (10) Michael G. Safonov, Michael Athans	(15) AF	8. CONTRACT OR GRANT NUMBER(s) AFOSR ~~77~~-3281-77, NGL-22-009-124
9. PERFORMING ORGANIZATION NAME AND ADDRESS Massachusetts Institute of Technology, Electronic Systems Laboratory, Cambridge, Massachusetts 02139	(16)	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F (17) 9769-01
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332	(11)	12. REPORT DATE Apr[redacted]77
		13. NUMBER OF PAGES 10
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	(12) 11p.	15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

This paper has been submitted to the 1977 IEEE Conference on Decision and Control and to the IEEE Transactions on Automatic Control.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Robustness properties of nonlinear extended Kalman filters with constant gains and modelling errors are presented. Sufficient conditions for the non-divergence of state estimates generated by such nonlinear estimators are given. In addition, the overall robustness and stability properties of closed-loop stochastic regulators, based upon the Linear-Quadratic-Gaussian design methodology using linearized dynamics, are presented; the sufficient conditions for closed-loop stability have a "separation-type" property.

The aforementioned CGEKF robustness, nondivergence, and regulator stability results are derived in the general context of the class of constant-gain nonlinear estimators whose design is not necessarily based on statistical considerations — for example, designs intended to optimize structural simplicity or error-transient response, i.e. nonlinear observers (cf. [4]). This general class of constant-gain nonlinear estimators includes as a special case the CGEKF, which is designed to be optimal with respect to a statistical criterion. One can suboptimally synthesize constant-gain nonlinear estimators of this general class employing the same design approach as is used for the CGEKF — using a constant linear approximation of the actual nonlinear system, simply design the residual-gain to be optimal with respect to the design criterion of interest. In the context of this broader class of suboptimal nonlinear estimators, our results provide analytically verifiable conditions which can be used to test the nondivergence of these estimators and to evaluate their robustness against the effects of design approximations; though one cannot in general expect such designs to be as robust as the CGEKF. The CGEKF output-feedback separation-type property extends to this broader class of estimators, showing that the nondivergent estimates can, unconditionally, be substituted true values in otherwise-stable feedback systems without ever causing instability.

2. RELATED LITERATURE

The literature on the subject of robustness and computational considerations in nonlinear estimation is sparse and largely inconclusive. The discussion of nonlinear estimation in Schweppe [5, ch. 13] provides a good intuitive understanding of the trade-offs between computational requirements and residual-gain choice; though the possibility of a constant residual-gain is not explicity considered. The idea of using a constant residual-gain for linear filtering is wellknown (cf. [1, pp. 238-242]), but the connection with nonlinear filtering has not been established. Of the existing literature on nonlinear estimation, [4] and [6] appear to be the most closely related to the present paper.

Gilman and Rhodes [6] suggest a procedure for synthesizing nonlinear estimators with a pre-computable, but time-varying, residual-gain. Their estimator, like the EKF and CGEKF, has the intuitively appealing structure of a model-reference estimator (cf. [5, p.403]); that is, it consists of an internal model of the system dynamics with observations entering via a gain acting on the residual error between the system and model outputs. The distinguishing feature of the estimator suggested in [6] is that the residual-gain is chosen so as to minimize a certain upper bound on the mean-square estimate error. This procedure tends to ensure a robust design since, assuming the minimal value of the error-bound does not "blow-up", the estimator cannot diverge. A limitation of this design procedure is that the error-bound may be very loose for systems with substantial nonlinearity; so there is no assurance that the bound-minimizing residual-gain is a good choice. Also, there is no a priori guarantee that the resultant estimator will even be stable since the minimal error-bound may become arbitrarily large as time elapses.

Tarn and Rasis [4] have proposed a constant-gain model-reference-type nonlinear estimator which is a natural extension of Luenberger's observer for linear systems, having a design based solely on stability considerations. The results of [4] show that, given such a nonlinear observer design, if certain Lyapunov functions can be found, then one can conclude that

a) The estimator is nondivergent;

b) The estimator can be used for state reconstruction in a full-state feedback system without causing instability. However, from an engineering standpoint the results of [4] are deficient in that they are nonconstructive: no design synthesis procedure is suggested; no method is proposed for constructing the Lyapunov functions required to test the stability of a design; no procedure is suggested for optimizing the estimate accuracy of the design. The CGEKF results presented in the present paper address all these deficiencies by providing a constructive procedure for synthesizing stable constant-gain model-reference estimator designs which are to a first approximation optimally accurate. Moreover, our results prove that, provided the estimator is nondivergent, it can be used for state reconstruction without ever causing instability, independent of the availability of any Lyapunov functions.

3. NOTATION AND TERMINOLOGY

In this paper the input-output view of systems is taken, considering a system to be an interconnection of "black boxes" each representable by its input-output characteristics. As will become apparent, the input-output view provides a convenient and natural setting for the discussion and analysis of estimator robustness and divergence, as well as feedback system stability. In this section the pertinent terminology drawn from [7]-[11] is reviewed and the notion of estimator divergence is formalized.

An operator is a mapping of functions into functions — such as is defined by a "black box" which maps input time-functions into output time-functions. An operator is said to be nonanticipative if the value assumed by its output function at any time instant t_0 does not depend on the values assumed by its input function at times $t > t_0$. An operator is said to be memoryless or equivalently nondynamical if the instantaneous value of its output at time t_0 depends only on the value of its input at time t_0. A dynamical operator is an operator which is not necessarily nondynamical.

To facilitate the discussion, the various input and output functions considered in this paper are presumed to be imbedded in function spaces of the type

$$M_2(R_+, R^r) \triangleq \{\underline{z}: R_+ \to R^r\} \qquad (3.1)$$

[11, p. 125] on which are defined the following inner product and norm respectively

$$\langle \underline{z}_1, \underline{z}_2 \rangle \triangleq \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} \underline{z}_1^T(t)\, \underline{z}_2(t)\,dt \tag{3.2}$$

$$\| \underline{z} \| \triangleq \sqrt{\langle \underline{z}, \underline{z} \rangle} \; . \tag{3.3}$$

The quantity $\| \underline{z} \|^2$ can be viewed as the "average power" in the function $\underline{z}$; in fact, if $\underline{z}$ is generated by a stationary random process (cf. [12, p. 300]), then $\| \underline{z} \|^2$ is simply the expected value of $\underline{z}^T(t)\, \underline{z}(t)$.

Because the space M_2 may be unfamiliar to many readers, we briefly discuss its relation to the similar, but distinct, space L_2 which is more widely used in input-output system analysis. The feature that distinguish M_2 from L_2 is the introduction of the "normalizing factor" $1/\tau$ into the inner product (3.2). Whereas, the L_2-norm is appropriately viewed as a measure of the "total energy" of a function, the normalizing factor $1/\tau$ leads to the "average power" interpretation of the norm (3.3). The space M_2 is larger than L_2, every function in L_2 being included in the subspace of M_2 comprised of functions of zero norm.

The gain or norm of an operator $\underset{\sim}{F}$, denoted $g(\underset{\sim}{F})$ and $\| \underset{\sim}{F} \|$ respectively, are defined by

$$g(\underset{\sim}{F}) \triangleq \| \underset{\sim}{F} \| \triangleq \sup_{0<\| \underline{z} \| <\infty} \frac{\| \underset{\sim}{F}\underline{z} \|}{\| \underline{z} \|} \; . \tag{3.4}$$

The incremental gain of $\underset{\sim}{F}$ is

$$\tilde{g}(\underset{\sim}{F}) \triangleq \sup_{0<\| \underline{z}_1 - \underline{z}_2 \| <\infty} \frac{\| \underset{\sim}{F}\underline{z}_1 - \underset{\sim}{F}\underline{z}_2 \|}{\| \underline{z}_1 - \underline{z}_2 \|} \; . \tag{3.5}$$

If $g(\underset{\sim}{F}) < \infty$, $\underset{\sim}{F}$ is said to have finite gain. Likewise, if $\tilde{g}(\underset{\sim}{F}) < \infty$, then $\underset{\sim}{F}$ is said to have finite incremental gain. The operator $\underset{\sim}{F}$ is bounded if inputs of finite norm produce outputs of finite norm; i.e., for each $\underline{z}$ with $\|\underline{z}\| < \infty$, there exists a scalar $\rho(\underline{z}) \geq 0$ such that $\| \underset{\sim}{F}\, \underline{z} \| < \rho(\underline{z})$. A dynamical system is said to be stable if the operator describing its input-output characteristics is bounded; the system is said to be finite gain stable if the operator has finite gain. An operator $\underset{\sim}{F}$ is said to be strongly positive, denoted $\underset{\sim}{F} > 0$, if for some $\varepsilon>0$ and all $\underline{z}$

$$\langle \underline{z}, \underset{\sim}{F}\, \underline{z} \rangle \geq \varepsilon \| \underline{z} \|^2 \; . \tag{3.6}$$

The operator $\underset{\sim}{F}$ is said to be positive, denoted $\underset{\sim}{F} \geq 0$, if (3.6) holds with $\varepsilon=0$.

The derivative of the operator $\underset{\sim}{F}$ at the point $\underline{z}_0$ is defined to be the linear operator $\nabla\underset{\sim}{F}(\underline{z}_0)$ having the property that for all $\underline{z}$

$$\nabla\underset{\sim}{F}(\underline{z}_0)\underline{z} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(\underset{\sim}{F}(\underline{z}_0 + \varepsilon\underline{z}) - \underset{\sim}{F}\underline{z}_0\right), \tag{3.7}$$

provided that the indicated limit exists. When the derivative $\nabla\underset{\sim}{F}(\underline{z}_0)$ exists, $\underset{\sim}{F}$ is said to be differentiable at $\underline{z}_0$. For example, if $\underset{\sim}{F}$ is memoryless, i.e., if $(\underset{\sim}{F}\underline{z})(t) = \underline{f}(\underline{z}(t))$ for some $\underline{f}: R^{r_1} \to R^{r_2}$, then $\nabla\underset{\sim}{F}(\underline{z})$ is simply the Jacobian matrix $\partial\underline{f}/\partial\underline{z}$ (cf. [8, p. 19]). Alternatively, if $\underset{\sim}{F}$ is a linear operator then $\nabla\underset{\sim}{F}(\underline{z}) = \underset{\sim}{F}$ for all $\underline{z}$.

The relevance of the above terminology to estimation stems from the fact that for each control input function $\underline{u}$, the error $\underline{e} \triangleq \hat{\underline{x}} - \underline{x}$ of an estimator can be represented as the output of an operator $\underset{\sim}{E}_{\underline{u}}$ whose inputs are the system and measurement noises, i.e.,

$$\underline{e} \triangleq \hat{\underline{x}} - \underline{x} = \underset{\sim}{E}_{\underline{u}}(\underline{\xi}, \underline{\theta}). \tag{3.8}$$

To formalize the notion of estimator divergence the following definitions are introduced: an estimator is nondivergent if its error operator is bounded uniformly in $\underline{u}$, i.e. if there exists a scalar $\rho(\underline{\xi}, \underline{\theta}) \geq 0$ such that

$$\sup_{\underline{u}} \| \underset{\sim}{E}_{\underline{u}}(\underline{\xi}, \underline{\theta}) \| \leq \rho(\underline{\xi}, \theta); \tag{3.9}$$

It is convergent if $\rho(\underline{\xi}, \underline{\theta}) \equiv 0$; it is nondivergent with finite gain if

$$\sup_{\underline{u}} g(\underset{\sim}{E}_{\underline{u}}) < \infty \, . \tag{3.10}$$

Evidently, convergence implies nondivergence with finite gain which in turn implies nondivergence. These definitions can be loosely interpreted as follows: an estimator is nondivergent if mean-square bounded disturbances produce mean-square bounded estimate-error; it is nondivergent with finite gain if the mean-square estimate-error is proportional to the magnitude of the disturbances; it is convergent if the mean-square error always tends to zero. An estimator that is not nondivergent is said to be divergent.

4. PROBLEM FORMULATION

We consider the problem of estimation for the nonlinear system

$$\left.\begin{aligned} \frac{d}{dt}\underline{x} &= \underset{\sim}{A}\underline{x} + \underset{\sim}{B}\underline{u} + \underline{\xi} \\ \underline{y} &= \underset{\sim}{C}\underline{x} + \underline{\theta} \end{aligned}\right\} \tag{4.1}$$

where

$\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$ are nonanticipative, continuous, almost-everywhere differentiable, dynamic nonlinear operators with finite incremental gain;

$\underline{\xi} \in M_2(R_+, R^n)$, $\underline{\theta} \in M_2(R_+, R^p)$ are disturbance input functions;

$\underline{y}$ is an R^p-valued observed output function;

$\underline{u}$ is an R^m-valued known control input function;

$\underline{x}$ is an R^n-valued function which is to be estimated based on knowledge of $\underline{y}$ and $\underline{u}$.

As a candidate for estimator we consider the constant-gain model-reference-type estimator

$$\left.\begin{aligned} \frac{d}{dt}\hat{x} &= A\,\hat{x} + B\,u - H(\hat{x})(\hat{y}-y) \\ \hat{y} &= C\,\hat{x} \end{aligned}\right\} \quad (4.2)$$

where $H(\hat{x})$ is a matrix of appropriate dimensions whose entries depend nondynamically on $\hat{x}$. When $H(\hat{x})$ is independent of $\hat{x}$ and when A and C are nondynamical, then (4.2) is identical in structure to the so-called observer for nonlinear stochastic systems proposed by Tarn and Rasis [4]; consequently we refer to the structure (4.2) as a <u>nonlinear observer</u>. The CGEKF is a type of nonlinear observer, just as a Kalman filter is a type of linear observer.

A useful method for describing the dynamical evolution of the nonlinear observer's error,

$$e \triangleq \hat{x} - x, \quad (4.3)$$

is by the feedback equations (see Fig. 1)

$$\left.\begin{aligned} \frac{d}{dt} e &= \tilde{A}(x)e + v \\ r &= \tilde{C}(x)e \end{aligned}\right\} \quad (4.4)$$

$$v \triangleq -H(\hat{x})\cdot(r - \theta) - \xi \quad (4.5)$$

where

$$r \triangleq \hat{y} - C\,x \quad (4.6)$$

and $\tilde{A}(x)$ and $\tilde{C}(x)$ are dynamical nonlinear operators defined by

$$\tilde{A}(x)z \triangleq A(x + z) - A\,x \quad (4.7)$$

$$\tilde{C}(x)z \triangleq C(x + z) - C\,z \quad (4.8)$$

for all $z \in M_2(R_+, R^n)$. From this feedback representation of the error dynamics of the nonlinear observer (4.2), it is immediately apparent that the problem of choosing the residual-gain $H(\hat{x})$ so as to make the estimator nondivergent is identical to the problem of choosing a stabilizing feedback for the system (4.4).

In order to facilitate the selection of a suitable residual-gain $H(\cdot)$, we assume that equation (4.4) describing the "open-loop error-dynamics" admits the nominal linearization

$$\left.\begin{aligned} \frac{d}{dt} e &= A(\hat{x})e + v \\ r &= C(\hat{x})e \end{aligned}\right\} \quad (4.9)$$

where $A(\hat{x})$ and $C(\hat{x})$ are matrices of appropriate dimensions whose entries in general depend nondynamically on $\hat{x}$. The idea is to choose the gain $H(\hat{x})$ assuming that $\hat{x}$ is a constant function and that the linearization (4.9) is exact. The problem is thus reduced to a time-invariant linear estimation problem for which several methods are available for choosing $H(\cdot)$, e.g. pole assignment [13, § 7.4] or Kalman filtering [14, Ch. 7]. For brevity of notation, the arguments of A, C, and H are suppressed in the sequal — clearly, the simplest estimator structure results when A, C, and hence H are chosen to be independent of $\hat{x}$.

5. NONLINEAR OBSERVER RESULTS

We now state two basic theorems concerning the nonlinear observer (4.2). The first result, Theorem 1, states that substitution of estimates generated by a nondivergent nonlinear observer for true values in an otherwise-stable feedback control system can never destabilize the closed-loop system. This result has obvious implications regarding the utility of nonlinear observers for state reconstruction in nonlinear optimal and suboptimal feedback control systems. The second result, Theorem 2, gives sufficient conditions for a nonlinear observer to be nondivergent. The proofs are in the appendix.

<u>Theorem 1</u>: Let G be a nonanticipative nonlinear dynamical operator with finite incremental gain. Suppose that the system (4.1) is closed-loop stable (finite gain stable) with feedback $u = G\,x$. Then the system (4.1) with feedback $u = G\,\hat{x}$ will also be closed-loop stable (finite gain stable).

<u>Theorem 2</u>: Let the matrices $P \equiv P(\hat{x}(t))$ and $S \equiv S(\hat{x}(t))$ be symmetric positive definite solutions of the Lyapunov equation

$$(A-HC)P + P(A-HC)^T + S = 0 \ . \quad (5.1)$$

If uniformly almost-everywhere

$$[(A-\nabla A(x))-H(C-\nabla C(x))]P + \frac{1}{2} S > 0 \ , \quad (5.2)$$

then the nonlinear observer (4.2) is nondivergent with finite gain.[1]

We note that the condition (5.1) of Theorem 2 is not restrictive: it can always be satisfied when the residual-gain H is chosen under the assumption the linearization (4.9) is exact since, for any given positive definite S, the Lyapunov equation (5.1) has a unique positive definite solution P if any only if $A-HC$ is a stable matrix [13, p.341]. In the situation in which $(A, C) = (A, C)$, the observer (4.2) will be nondivergent if and only if $(A-HC)$ is stable. Thus, when the residual-gain H is chosen under the assumption that the linearization (4.9) is exact, a P and an S satisfying (5.1) can always be found by simply picking any positive definite S and solving (5.1) for P.

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[1] In general the matrices A, C, H, P, and S depend on $\hat{x}(t)$. In this case (5.2) must hold <u>uniformly</u> with respect to $\hat{x}$. Consequently, allowing A, C, H, P and S to depend on $\hat{x}$ offers no advantage with regard to satisfying the nondivergence conditions of Theorem 2; though it may improve the estimator's error statistics.

The interesting part of Theorem 2 is the condition (5.2). It characterizes a class of nonlinearities for which the nonlinear observer (4.2) is assured of being nondivergent. An important feature of Theorem 2 is the form of condition (5.2)—it is expressed in terms of the deviation of the system (4.4) from the linearization (4.9) used in selecting the residual-gain. When the deviation is zero (i.e., $(A, C) \equiv (A, C)$) then the condition (5.2) is always satisfied since S is positive definite.

The question naturally arises "How difficult is it to verify condition (5.2)?" The fact that the left-hand side of (5.2) is linear in A and C and the fact that a positively-weighted sum of positive operators is positive makes (5.2) much easier to verify than might be apparent at first inspection. For example, if A and C are memoryless and if there are constants $c_{ij}^{(\ell)}$, $a_{jk}^{(\ell)}$ ($\ell = 1, 2$; $i = 1, \ldots, p$; $j,k=1, \ldots, n$) such that for all $x \in R^n$

$$0 \geq c_{ij}^{(1)} \leq [C - \nabla C(x)]_{ij} \leq c_{ij}^{(2)} \geq 0 \tag{5.3}$$

$$0 \geq a_{jk}^{(1)} \leq [A - \nabla A(x)]_{jk} \leq a_{jk}^{(2)} \geq 0 \tag{5.4}$$

(where $[M]_{ij}$ denotes the ij-th element of the matrix M), then one may readily verify that sufficient conditions for (5.2) to hold are

$$c_{ij}^{(\ell)}(P\, e_j\, e_i^T H^T + H\, e_i\, e_j^T P) + S > 0 \tag{5.5}$$

$$a_{jk}^{(\ell)}(P\, e_k\, e_j^T + e_j\, e_k^T P) + S > 0 \tag{5.6}$$

(where e_i denotes the i-th standard basis vector, i.e., the vector whose elements are zero except the i-th which is a one). To verify conditions (5.5)-(5.6) requires that one check the positive definiteness of as many nxn-matrices as there are nonzero elements in the set

$$\{c_{ij}^{(\ell)}, a_{jk}^{(\ell)} \,|\, \ell=1,2;\ i=1, \ldots, p;\ j,k=1, \ldots, n\}$$

(which can be done, for example, by checking that the principal leading minors of each matrix are positive [13, p. 341]). So, if the nonlinear system (4.4) is identical to the linearization (4.9) except for N memoryless nonlinearities, then one need only check the positive definiteness of at most 2N nxn-matrices to verify (5.2).

6. THE CONSTANT-GAIN EXTENDED KALMAN FILTER (CGEKF)

Intuitively, it is clear that if the linearization (4.9) is sufficiently faithful to the nonlinear system (4.4), then the error response of the nonlinear observer (4.2) will be close to the error response one would get in the ideal situation in which the linearization is exact. This intuition is validated by the error-bounding results of [6], [15]. Consequently, if the disturbances ξ and θ are reasonably well approximated by zero-mean white noise, then it is reasonable to expect that a good suboptimal minimum variance estimator can be obtained by choosing the residual-gain H to be the minimum-variance-optimal gain for the linearized system (4.9), i.e., the Kalman filter gain [14, p. 214]

$$H = \Sigma\, C^T\, \Theta^{-1} \tag{6.1}$$

where $\Sigma = \Sigma^T > 0$ satisfies the Riccati equation[2]

$$0 = \Sigma\, A^T + A\, \Sigma - \Sigma\, C^T\, \Theta^{-1}\, C\, \Sigma + \Xi \tag{6.2}$$

and Ξ and Θ are the positive definite covariance matrices of the disturbances ξ and θ respectively.[3] The resultant estimator is the constant-gain extended Kalman filter (CGEKF) depicted in Fig. 2.

A surprising and important consequence of the CGEKF approach to nonlinear observer design is that, in addition to yielding a suboptimally accurate estimator design, the CGEKF design procedure is inherently robust in the sense that even a crude linearization (4.9) will suffice for residual-gain design. The CGEKF design procedure automatically ensures that the deviation from the design linearization admissible under the conditions of Theorem 2 can be quite large. The extent of this robustness is quantified in the following result:

Theorem 3 (CGEKF Robustness): If uniformly almost-everywhere

$$[(A-\nabla A(x)) - H(C-\nabla C(x))]\Sigma + \frac{1}{2}(\Xi + \Sigma\, C^T \Theta^{-1} C\, \Sigma) > 0, \tag{6.3}$$

then the CGEKF is nondivergent with finite gain.[4]

Proof: Let

$$S = \Xi + \Sigma\, C^T\, \Theta^{-1}\, C\, \Sigma \tag{6.4}$$

$$P = \Sigma .$$

Then (6.2) and (6.3) ensure that (5.1) and (5.2) respectively are satisfied. The result follows from Theorem 2.

To fully appreciate the implications of Theorem 3 with regard to the robustness of the CGEKF design

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[2] We assume that the required controllability and observability conditions are satisfied so that there is a unique positive definite solution of (6.2) (cf. [14, pp. 234-243]).

[3] In general, Θ, Ξ, A, and C may be chosen to be dependent on $\hat{x}(t)$, in which case Σ and H also depend on $\hat{x}(t)$.

[4] As with Theorem 2, no advantage with regard to satisfying the nondivergence conditions of Theorem 3 results from choosing Θ, Ξ, A, or C to be dependent on $\hat{x}$ (cf. footnote 1).

procedure, it is instructive to consider the situation in which for all $\underset{\sim}{x}$

$$\underset{\sim}{A}(\underline{x}) = \underline{A} \tag{6.6}$$

$$\underset{\sim}{C}(\underline{x}) = [\text{diag}\,(\underset{\sim}{N}_1, \ldots, \underset{\sim}{N}_p)]\underline{C} \tag{6.7}$$

so that all the differences between the open-loop error dynamics system (4.4) and the design linearization (4.9) are lumped into the p dynamical nonlinearities, $\underset{\sim}{N}_i$ (i=1, ..., p), which are in series with the system outputs. This is equivalent to all nonlinearity in the system (4.1) being lumped in the actuators and sensors (see Fig. 4). It is emphasized that this does not mean that we are restricting our attention to systems with only actuator and sensor nonlinearity; rather, we are merely stipulating that the actual system's open-loop error dynamics have the same input-output behavior as such a system.

For simplicity, we further assume $\underline{\Theta}$ is of the form

$$\underline{\Theta} = \text{diag}\,(\theta_{11}, \theta_{22}, \ldots, \theta_{pp})\,. \tag{6.8}$$

With (6.6)-(6.8) satisfied, the nondivergence condition (6.3) of Theorem 3 reduces to

$$\underline{\Sigma}\underline{C}^T\,\text{diag}[\theta_{11}^{-1}(\nabla\underset{\sim}{N}_1(\underline{x})-\tfrac{1}{2}), \ldots, \theta_{pp}^{-1}(\nabla\underset{\sim}{N}_p(\underline{x})-\tfrac{1}{2})]\underline{C}\underline{\Sigma}$$

$$+\frac{1}{2}\underline{\Xi} > 0, \tag{6.9}$$

which is satisfied if

$$\nabla\underset{\sim}{N}_i(\underline{x}) \geq \frac{1}{2} \qquad (i=1, \ldots, p)\,. \tag{6.10}$$

The condition (6.10) establishes a "lower bound" the inherent robustness of the CGEKF design procedure, i.e., every CGEKF design can tolerate at least nonlinearities satisfying (6.10). One can interpret this inherent robustness in terms of the gain and phase margin of the feedback representation (cf. Fig. 3) of the CGEKF error dynamics as follows: Suppose that the $\underset{\sim}{N}_i$ (i=1, ..., p) are linear dynamical elements with respective transfer functions $L_i(s)$ (i=1, ..., p). Then, condition (6.10) becomes

$$\text{Re}[L_i(j\omega)] \geq \frac{1}{2} \quad (i=1, \ldots, p), \tag{6.11}$$

i.e. the Nyquist locus of each $L_i(j\omega)$ must lie to the right of the vertical line in the complex plane passing through the point $\frac{1}{2} + j0$. For example, if $L_i(s)$ (i=1, ..., p) are nondynamical linear gains, i.e., $L_i(j\omega)=k$, then (6.11) becomes

$$k \geq \frac{1}{2}\,. \tag{6.12}$$

Alternatively, if

$$L_i(s) = e^{j\phi_i} \qquad (i=1, \ldots, p)$$

corresponding to a pure phase shift of angle ϕ_i (i=1, ..., p) in the p respective output channels of the open-loop error dynamics system, then condition (6.11) becomes

$$|\phi_i| \leq 60°. \tag{6.13}$$

One can interpret the conditions (6.12) and (6.13) as saying the CGEKF design procedure leads to an infinite gain margin, at least 50% gain reduction tolerance, and at least ± 60° phase margin in each output channel of the error dynamics feedback system (Fig. 3) — the margins being relative to the ideal situation in which the linearization (4.9) is exact. Engineers experienced in classical servomechanism design will recognize that these minimal stability margins are actually quite large, ensuring that the nonlinear observer error dynamics feedback system of Fig. 3 will be stable despite substantial differences between the design linearization (4.9) and the system (4.4). Consequently, the CGEKF design procedure is assured of yielding a nondivergent nonlinear observer design for systems with a good deal of nonlinearity.

This surprisingly large robustness of the CGEKF design procedure is mathematically dual to the robustness of linear-quadratic state-feedback regulators reported in [2]-[3], wherein full-state-feedback linear optimal regulators are shown to have infinite gain margin, 50% gain reduction tolerance, and ± 60° phase margin in each input channel. This duality, which is a consequence of the symmetry between the equations governing the regulation error of linear optimal regulators and the equations governing the estimate error of the CGEKF (cf. eqns. (B.1) and (4.3) of [2] vs. eqns. (4.4) and (6.2) here), provided the principal source of inspiration for the work leading to these robustness results.

7. PRACTICAL CGEKF SYNTHESIS

The results of the preceding section provide a basis for computed-aided-design of practical, nondivergent CGEKF estimators. The following procedure shows how these results might be employed for this purpose:

Step 1

Pick constant values for $\underline{A}$, $\underline{C}$, $\underline{\Xi}$ and $\underline{\Theta}$. The values of $\underline{A}$ and $\underline{C}$ should be initially chosen to reflect as closely as possible the derivatives $\nabla\underset{\sim}{A}(\underline{x})$ and $\nabla\underset{\sim}{C}(\underline{x})$ respectively, i.e., so that $\|\underline{A}-\nabla\underset{\sim}{A}(\underline{x})\|$ and $\|\underline{C}-\nabla\underset{\sim}{C}(\underline{x})\|$ are small, at least for those values of $\underline{x}$ which are most probable — statistical linearization methods (cf. [16, Ch. 7]) may be helpful in this regard. The matrices $\underline{\Theta}$ and $\underline{\Xi}$ should be initially chosen to reflect the covariance of the disturbances $\underline{\theta}$ and $\underline{\xi}$ respectively. If the input-output relations of the operators $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ are not precisely known, then the designer may wish to consider compensating for this using state-augmentation following the spirit of [17]-[18] in order to reduce bias errors.

Step 2

Compute $\underline{\Sigma}$ and $\underline{H}$ from (6.1) and (6.2). This can be done with the aid of a digital computer using available software for solving the Riccati equation.

Step 3

Test the resultant CGEKF design for nondivergence. This can be done any of the following ways:

a) By checking the conditions of Theorem 3;
b) By direct digital Monte Carlo simulation;
c) By approximate describing-function simulation [1, § 6.4].

If the estimator is divergent, go to Step 5; otherwise, procede to Step 4.

Step 4

Check the nondivergent CGEKF for satisfactory performance, i.e., for acceptable error statistics. This can be done using one or more of the following approaches:

a) By direct Digital Monte Carlo Simulation;
b) By approximate describing-function simulation [1, § 6.4];
c) By using the error-bounding results in [6], [15].

If performance is acceptable, stop. Otherwise, introduce estimate dependent matrices $\underline{A}(\hat{\underline{x}}(t))$, $\underline{C}(\hat{\underline{x}}(t))$ so as to further reduce $\|\underline{A}(x)-\nabla\tilde{A}(\underline{x})\|$ and $\|\underline{C}(\underline{x}) - C(\underline{x})\|$; as in Step 1 statistical linearization methods may be help here. Compute $\underline{\Sigma}(\hat{\underline{x}})$ and $\underline{H}(\hat{\underline{x}})$ from (6.1) and (6.2) and return to Step 3.

Step 5

Take the divergent CGEKF and, assisted by a computer, determine the values of $\underline{x}$ for which the condition (6.3) is not satisfied. Modify the matrices $\underline{A}$ and $\underline{C}$ so as to reduce the magnitude $\|\underline{A}- \nabla A(x)\|$ and $\|\underline{C} - \nabla C(\underline{x})\|$ at these values of $\underline{x}$. If necessary, adjust the $\underline{\Xi}$ and $\underline{\Theta}$ matrices. Return to Step 3.

For systems that are not "too nonlinear" this procedure can be expected to converge rapidly to an acceptable CGEKF design. However, for highly nonlinear systems, the procedure may not lead easily to a satisfactory design, even when such a design exists. A noteworthy limitation of the procedure is that no explicit method is provided for selecting the "best" modifications of $\underline{A}$, $\underline{C}$, $\underline{\Xi}$, and $\underline{\Theta}$ as required in Step 5.

Even in cases where a nondivergent CGEKF estimator is not possible, it may be possible to exploit Theorem 3 to construct a CGEKF estimator which, if properly initialized and if not subjected to excessively large disturbances, has satisfactory performance. This is accomplished by using estimate-dependent matrices $\underline{A}(\hat{\underline{x}})$, $\underline{C}(\hat{\underline{x}})$, $\underline{\Theta}(\hat{\underline{x}})$, and $\underline{\Xi}(\hat{\underline{x}})$ so that $\underline{H}\equiv\underline{H}(\hat{\underline{x}})$ and $\underline{\Sigma}\equiv\underline{\Sigma}(\hat{\underline{x}})$ become estimate dependent. Provided that the estimate error $\underline{e}\triangleq\underline{x}-\underline{x}$ remains small enough so that $\underline{x}$ lies in the region in which (6.3) is satisfied, then the estimator cannot diverge. It is emphasized that such an estimator requires careful initialization and may not be able to recover from large disturbances without re-initialization, much like the EKF which in general has similar limitations.

8. SUBOPTIMAL NONLINEAR OUTPUT-FEEDBACK CONTROLLERS

The CGEKF results of the present paper combine with the results of [2]-[3] on the nonlinearity tolerance of linear-quadratic state-feedback (LQSF) control laws to suggest a simple, practical nonlinear extension of the celebrated linear-quadratic-Gaussian optimal output-feedback control design technique. The idea is to cascade a CGEKF estimator with a constant LQSF gain matrix, both optimally designed for the time-invariant nominal linearization of the system (4.1)

$$\begin{aligned} \dot{\underline{x}} &= \underline{A}\,\underline{x} + \underline{B}\,\underline{u} + \underline{\xi} \\ \underline{y} &= \underline{C}\,\underline{x} + \underline{\theta} \end{aligned} \tag{8.1}$$

with performance index

$$J(\underline{x}, \underline{u}) \triangleq E\,[\lim_{\tau\to\infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} \underline{x}^T(t)\underline{Q}\,\underline{x}(t) + \underline{u}^T(t)\underline{R}\,\underline{u}(t)dt] \tag{8.2}$$

where

$\underline{\xi}$ and $\underline{\theta}$ are zero-mean white Gaussian with respective covariance matrices $\underline{\Xi}$ and Θ;

$\underline{A}$, $\underline{B}$, $\underline{C}$ are matrices of appropriate dimensions;

$\underline{R}$, $\underline{Q}$ are positive definite weighting matrices of appropriate dimensions.

In general $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{R}$, $\underline{Q}$, $\underline{\Theta}$, and $\underline{\Xi}$ may be chosen to be nondynamically dependent on $\hat{\underline{x}}$. Assuming that the linearization (8.1) is exact, the optimal Kalman filter residual gain is given by (6.1)-(6.2) and the optimal LQSF feedback is given by

$$\underline{u} = -\underline{R}^{-1}\,\underline{B}^T\,\underline{K}\,\underline{x} \tag{8.3}$$

where $\underline{K} = \underline{K}^T > \underline{0}$ satisfies the time-invariant Riccati equation

$$\underline{0} = \underline{K}\,\underline{A} + \underline{A}^T\underline{K} - \underline{K}\,\underline{B}\,\underline{R}^{-1}\,\underline{B}^T\underline{K} + \underline{Q}\,. \tag{8.4}$$

Cascading the CGEKF with the feedback (8.3) leads to the suboptimal nonlinear output-feedback control law (see Fig. 4)

$$\left.\begin{aligned} \underline{u} &= -\underline{R}^{-1}\,\underline{B}^T\underline{K}\,\hat{\underline{x}} \\ \frac{d}{dt}\hat{\underline{x}} &= \underset{\sim}{A}\,\hat{\underline{x}} + \underset{\sim}{B}\,\underline{u} - \underline{\Sigma}\,\underline{C}^T\,\underline{\Theta}^{-1}(\hat{\underline{y}}-\underline{y}) \\ \hat{\underline{y}} &= \underset{\sim}{C}\,\hat{\underline{x}}\,. \end{aligned}\right\} \tag{8.5}$$

This approach to suboptimal nonlinear output-feedback control design is similar in spirit to the approach outlined in [19], wherein an extended Kalman filter is cascaded with a time-varying suboptimal feedback gain; however the pre-computed constant gains in the control law (8.5) make it drastically simpler to implement from the standpoint of real-time computational burden. The remarkable robustness of the CGEKF design procedure and of LQSF control designs [2]-[3] assure that this approach will produce a stabilizing feedback control law for systems with even substantial nonlinearity. The extent of this robustness is quantified in the following result:

Theorem 4 (nonlinear output-feedback robustness): If uniformly almost-everywhere

$$[\underline{A} - \nabla \underset{\sim}{A}(\underline{x}) + (-\underline{\Sigma}\, \underline{C}^T\, \underline{\Theta}^{-1})\, (\underline{C} - \nabla \underset{\sim}{C}(\underline{x}))]\underline{\Sigma}$$

$$+ \frac{1}{2}\, (\underline{\Xi} + \underline{\Sigma}\, \underline{C}^T\, \underline{\Theta}^{-1}\, \underline{C}\, \underline{\Sigma}) > \underline{0} \tag{8.6}$$

and if

$$\underline{K}[\underline{A} - \underset{\sim}{A} + (\underline{B}-\underset{\sim}{B})\, (-\underline{R}^{-1}\underline{B}^T\underline{K})] + \frac{1}{2}(\underline{Q} + \underline{K}\, \underline{B}\, \underline{R}^{-1}\underline{B}^T\underline{K}) > \underline{0}\,, \tag{8.7}$$

then the system (4.1) with output-feedback (8.5) (as is depicted in Fig. 2) is finite gain stable.

Proof: This result is a direct consequence of Theorem B.1 of [2] and of Theorems 1 and 3 of this paper: Applying Theorem 3, condition (8.6) ensures that the CGEKF is nondivergent with finite gain; applying Theorem B.1 of [2], condition (8.7) ensures that the system (4.1) with full-state feedback (8.3) is stable with finite gain; the result follows from Theorem 1.

We emphasize that the matrices $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{Q}$, $\underline{R}$, $\underline{\Theta}$, $\underline{\Xi}$ in general can be chosen to depend on $\hat{\underline{x}}$. This may be helpful in optimizing the closed-loop transient response of the suboptimal nonlinear output-feedback system — especially, in adjustable set-point regulator designs, where it may be preferable to have a feedback law which is dependent on the system's operating point.

9. CONCLUSIONS

Efforts to find methods for reducing the real-time computational burden of the extended Kalman filter have led us to consider the possiblity of a constant-gain extended Kalman filter (CGEKF), designed to be optimal for a constant linear approximation of the actual nonlinear system. Since the residual-gain for a CGEKF estimator is constant and precomputable, the enormous real-time computational burden of error-covariance propagation and residual-gain updating is eliminated, drastically reducing real-time computational requirements. Because in many applications the linearization and disturbance modeling approximations made in CGEKF design may be only slightly cruder than the gross approximations that are made in EKF design, it is expected that the error-performance or CGEKF designs may actually be competitive with EKF designs in many applications.

By representing a nonlinear estimator as a servomechanism in which error is the output to be regulated, we have been able to apply modern input-output techniques of analysis to generate results explicitly characterizing the robustness of CGEKF estimators — and, more generally, estimators having the structure of the nonlinear observer (4.2) — against the effects of approximations introduced in designing the residual gain. These results have the form of analytically verifiable conditions on the deviation of the constant linear design model from the actual nonlinear system. The conditions, when satisfied, assure that the estimator is nondivergent. The conditions have been used to prove that the CGEKF design procedure is intrinsically robust in that the procedure automatically leads to a nondivergent estimator design for systems with even a relatively large degree of nonlinearity. The extent of this nonlinearity tolerance is quantitatively characterized by Theorem 3. The synthesis of practical CGEKF designs has been discussed and it has been shown that the CGEKF nondivergence conditions can be exploited to constructively modify and improve CGEKF designs.

A new method, based on linear-quadratic-Gaussian optimal feedback theory, has been proposed for the synthesis of suboptimal output-feedback control laws for nonlinear systems. The method leads to a simply-structured nonlinear dynamical feedback law that is drastically simpler to implement than suboptimal linear-quadratic-Gaussian nonlinear feedback controllers incorporating a time-varying gain and an EKF (cf. [19]). The feedback law decomposes naturally into an LQSF gain matrix and a CGEKF estimator in a fashion reminiscent of the way the separation theorem of estimation and control leads to a similar decomposition in linear problems. It has been shown that the inherent robustness of the CGEKF design procedure and of linear-quadratic state-feedback combine to assure that this design approach will lead to a stable feedback law for systems with substantial nonlinearity.

A limitation of the synthesis procedure for CGEKF estimators presented in this paper is that for highly nonlinear systems the effectiveness of the procedure becomes greatly dependent on the designer's intuition and judgement in selecting the linearization and noise covariance matrices ($\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{\Xi}$, $\underline{\Theta}$). While the use of (6.3) in Step 5 provides valuable guidance in this regard, it does not provide a clearly defined algorithm. This will be the topic of future research.

Regarding other future research possibilities, it is noteworthy that the assumption in this paper that the dependence of $\underline{H}$ on $\hat{\underline{x}}$ is memoryless is superfluous; that is, all of the conclusions, results, and proofs in this paper remain valid without this assumption. However, if $\underline{H}$ depends on $\hat{\underline{x}}$ in a more complex way — such as it does in the extended Kalman filter — it is generally much more difficult, if not impossible, to verify the nondivergence conditions of Theorems 2 and 3. Future research may reveal practical tests, similar to those introduced in this paper, for verifying the nondivergence of extended Kalman filters and other model-reference-type estimators having a residual-gain with a complex dynamical dependence on the estimate history.

10. ACKNOWLEDGEMENT

We gratefully acknowledge the assistance of Prof. S.K. Mitter in bringing to our attention the work of Corduneanu [11] on the function space $M_2(R_+, R^r)$.

APPENDIX

In this appendix the results of Zames [7] (as elaborated upon in [9]) are used to prove Theorems 1 and 2. We begin by introducing a definition.

Definition: Let $\underset{\sim}{F}: X \to Y$ be an operator. Then the incremental operator $\tilde{\underset{\sim}{F}}(\underline{x})$ is defined by

$$\tilde{\underset{\sim}{F}}(\underline{x})\underline{\delta x} \triangleq \underset{\sim}{F}(\underline{x} + \underline{\delta x}) - \underset{\sim}{F}\, \underline{x} \tag{A1}$$

for all $\underset{\sim}{x}$ and $\delta\underset{\sim}{x}$ elements of X.

Proof of Theorem 1:

$$\underset{\sim}{B}\,\underset{\sim}{G}\,\dot{\underset{\sim}{x}} = \underset{\sim}{B}\,\underset{\sim}{G}(\underset{\sim}{x}+\underset{\sim}{e}) = \underset{\sim}{B}\,\underset{\sim}{G}\,\underset{\sim}{x} + (\widetilde{\underset{\sim}{B}\,\underset{\sim}{G}})(\underset{\sim}{x})\,\underset{\sim}{e} \qquad \text{(A2)}$$

Let $\underset{\sim}{\xi}' \triangleq \underset{\sim}{\xi} + (\widetilde{\underset{\sim}{B}\,\underset{\sim}{G}})(\underset{\sim}{x})\underset{\sim}{e}$. Then the dynamics of the closed-loop system with $\underset{\sim}{u} = \underset{\sim}{G}\,\hat{\underset{\sim}{x}}$ satisfy

$$\frac{d}{dt}\underset{\sim}{x} = (\underset{\sim}{A} + (\underset{\sim}{B}\,\underset{\sim}{G}))\underset{\sim}{x} + \underset{\sim}{\xi}'. \qquad \text{(A3)}$$

whereas the dynamics with $\underset{\sim}{u} = \underset{\sim}{G}\underset{\sim}{x}$ satisfy

$$\frac{d}{dt}\underset{\sim}{x} = (\underset{\sim}{A} + \underset{\sim}{B}\,\underset{\sim}{G})\underset{\sim}{x} + \underset{\sim}{\xi}\,. \qquad \text{(A4)}$$

Since by hypothesis (A4) is stable (finite gain stable), it is sufficient to observe that

$$\|\underset{\sim}{\xi}'\| \triangleq \|\underset{\sim}{\xi} + (\widetilde{\underset{\sim}{B}\,\underset{\sim}{G}})(\underset{\sim}{x})\underset{\sim}{e}\| \le \|\underset{\sim}{\xi}\| + \|(\widetilde{\underset{\sim}{B}\,\underset{\sim}{G}})(\underset{\sim}{x})\underset{\sim}{e}\|$$

$$\le \|\underset{\sim}{\xi}\| + \tilde{g}(\underset{\sim}{B})\tilde{g}(\underset{\sim}{G})\|\underset{\sim}{e}\| < \infty. \qquad \text{(A5)}$$

Proof of Theorem 2

Let s denote the linear functional operator $s = \frac{d}{dt}$. From (4.4)-(4.5) it follows that

$$s\,\underset{\sim}{e} = (\tilde{\underset{\sim}{A}}(\underset{\sim}{x}) - \underset{\sim}{H}\,\tilde{\underset{\sim}{C}}(\underset{\sim}{x}))\underset{\sim}{e} - (\underset{\sim}{\xi} - \underset{\sim}{H}\,\underset{\sim}{\theta}). \qquad \text{(A6)}$$

Pre-multiplying by $\underset{\sim}{P}^{-1}$, introducing the dummy variable $\underset{\sim}{w}$ and the arbitrary constant $\varepsilon>0$, and rearranging yields

$$(\underset{\sim}{P}^{-1}\underset{\sim}{e}) = (s+\varepsilon)^{-1}\,\underset{\sim}{P}^{-1}\,\underset{\sim}{w} \qquad \text{(A7a)}$$

$$\underset{\sim}{w} = -(-\tilde{\underset{\sim}{A}}(\underset{\sim}{x}) + \underset{\sim}{H}\,\tilde{\underset{\sim}{C}}(x) - \varepsilon\underset{\sim}{I})\underset{\sim}{P}\,(\underset{\sim}{P}^{-1}\underset{\sim}{e}) - (\underset{\sim}{\xi} - \underset{\sim}{H}\,\underset{\sim}{\theta}). \qquad \text{(A7b)}$$

From Theorem 3 of [7] a sufficient condition for (A7) to be finite gain stable is the existence of an $\varepsilon>0$ such that[5]

$$(s+\varepsilon)^{-1}\underset{\sim}{P}^{-1} \ge \underset{\sim}{0} \qquad \text{(A8a)}$$

$$(-\tilde{\underset{\sim}{A}}(\underset{\sim}{x}) + \underset{\sim}{H}\tilde{\underset{\sim}{C}}(\underset{\sim}{x}) - \varepsilon\underset{\sim}{I})\underset{\sim}{P} > \underset{\sim}{0} \qquad \text{(A8b)}$$

uniformly for all $\underset{\sim}{x} \in M_2(R_+, R^n)$. Parseval's theorem ensures that (A8a) holds for all $\varepsilon > 0$. Define

$$\underset{\sim}{F} \triangleq (\underset{\sim}{A} - \tilde{\underset{\sim}{A}} - \underset{\sim}{H}(\underset{\sim}{C} - \tilde{\underset{\sim}{C}}))\underset{\sim}{P} + \frac{1}{2}\underset{\sim}{S}\,. \qquad \text{(A9)}$$

Then in view of (5.1), a necessary and sufficient condition for (A8b) to hold is

$$\tilde{\underset{\sim}{F}}(\underset{\sim}{x}) > \underset{\sim}{0} \qquad \text{(A10)}$$

uniformly for all $\underset{\sim}{x} \in M_2(R_+, R^n)$. Now, for all $\underset{\sim}{\eta} \in M_2(R_+, R^n)$.

$$\tilde{\underset{\sim}{F}}(\underset{\sim}{x})\underset{\sim}{\eta} \triangleq \underset{\sim}{F}(\underset{\sim}{x} + \underset{\sim}{\eta}) - \underset{\sim}{F}\,\underset{\sim}{\eta}$$

$$= \int_{\underset{\sim}{x}}^{\underset{\sim}{x}+\underset{\sim}{\eta}} \nabla\underset{\sim}{F}(\underset{\sim}{z})\,d\underset{\sim}{z}$$

$$= \int_0^1 \nabla\underset{\sim}{F}(\underset{\sim}{x}+\rho\,\underset{\sim}{\eta})\underset{\sim}{\eta}\,d\rho\,. \qquad \text{(A11)}$$

So, for all $\underset{\sim}{\eta} \in M_2(R_+, R^n)$

$$\langle\underset{\sim}{\eta}, \tilde{\underset{\sim}{F}}(\underset{\sim}{x})\underset{\sim}{\eta}\rangle = \langle\underset{\sim}{\eta}, \int_0^1 \nabla\underset{\sim}{F}(\underset{\sim}{x}+\rho\underset{\sim}{\eta})\underset{\sim}{\eta}\,d\rho\rangle$$

$$= \int_0^1 \langle\underset{\sim}{\eta}, \nabla\underset{\sim}{F}(\underset{\sim}{x}+\rho\underset{\sim}{\eta})\underset{\sim}{\eta}\rangle\,d\rho\,. \qquad \text{(A12)}$$

Thus, a sufficient condition for (A10) and hence (A8b) to hold is $\nabla\underset{\sim}{F}(\underset{\sim}{x})$ uniformly almost-everywhere strongly positive; that is, uniformly almost-everywhere

$$[(\underset{\sim}{A} - \nabla\underset{\sim}{A}(x)) - \underset{\sim}{H}(\underset{\sim}{C} - \nabla\underset{\sim}{C}(\underset{\sim}{x}))]\underset{\sim}{P} + \frac{1}{2}\underset{\sim}{S} > \underset{\sim}{0}. \qquad \text{(A13)}$$

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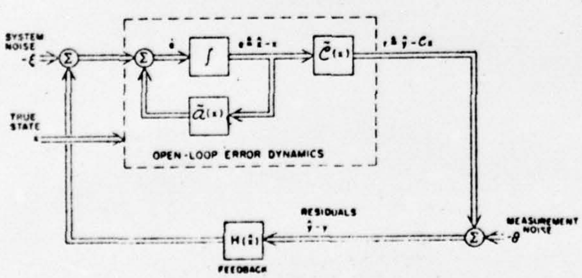

Fig. 1 Feedback Representation of the Error Dynamics of a Nonlinear Observer

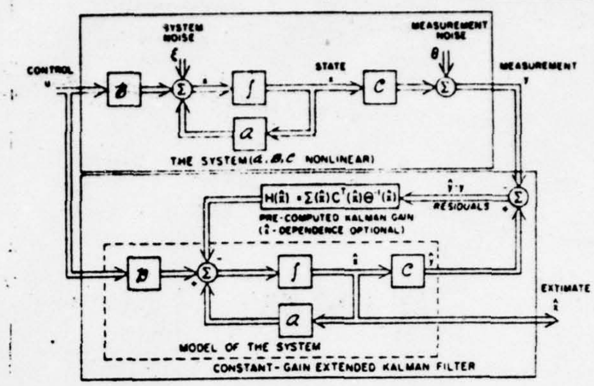

Fig. 2 Constant-Gain Extended Kalman Filter (CGEKF)

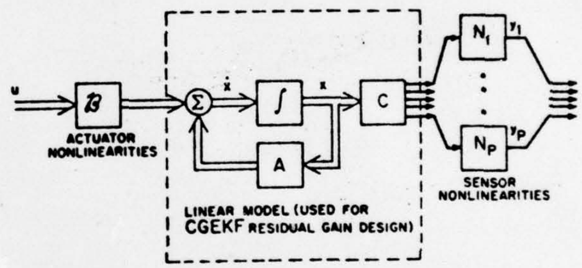

Fig. 3 System with All Nonlinearity Lumped in Actuators and Sensors

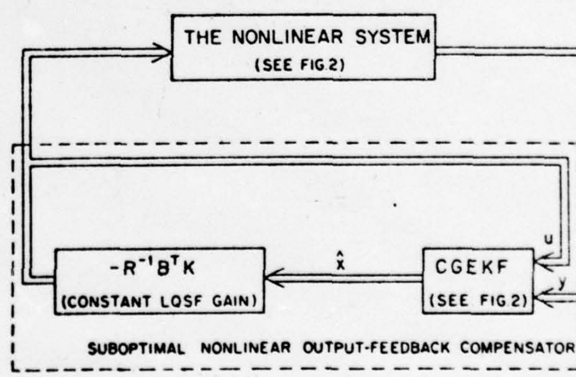

Fig. 4 Suboptimal Nonlinear Output-Feedback Compensator